

On many-valued modal logics and graded beliefs

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Abstract

Modal logics are well-known formalisms that account for representing and reasoning about notions such as necessity, belief, uncertainty, knowledge, similarity, obligations, time, etc. For instance Halpern's book [9] discusses a family of modal logics to reason about uncertainty. On the other hand, many-valued logical systems under the umbrella of the so-called *mathematical fuzzy logic* (in the sense of Hájek [3]) appear as suitable logical frameworks to formalize reasoning with gradual properties, i.e. notions whose satisfaction is a matter of degree. Therefore, if one is interested in reasoning involving both gradualness and some sort of modalities one is led to study systems of many-valued modal logic. In this talk we will mainly survey various families of many-valued or fuzzy modal logic of different nature that have been proposed in the literature to reason about different models of uncertainty in the sense of graded belief [6,8] and about preferences [12], both qualitatively and quantitatively. One of such families consist of two-layer modal systems, where the inner logic deals with the objects (events) over we quantify the uncertainty (that can be classical logic or other modal or many-valued logic), and the outer (many-valued) modal logic is used to reason about the involved modal notion itself (e.g. probabilistic or possibilistic uncertainty). These systems are relatively simple to axiomatise as they allow neither nested modalities nor mixed propositional and modal formulas. Typical and initial examples of these systems are fuzzy probabilistic logics where the outer logic is $[0, 1]$ -valued Łukasiewicz logic [7,8]. Other families of many-valued modal logics follow a more traditional approach, after Fitting's pioneering work [4], with a classical-like language extended with modal operators and with Kripke-style semantics, where possible worlds and the accessibility relation are many-valued, see e.g. [1,2]. Representative examples of these systems are Caicedo and Rodríguez's minimal modal logics based on Gödel $[0, 1]$ -valued logic [2], whose KD45-like extension has been recently shown to be adequate to reason about possibilistic uncertainty [10]. Finally, we will discuss different systems in this setting as well as some recent negative results by Vidal about non-axiomatizability of modal logics based on certain many-valued logics [11].

Acknowledgments

The author acknowledges partial support by the MOSAIC project (EU H2020- MSCA-RISE-2020 Project 101007627). and the PID2019-111544GB-C21 Spanish project ISINC funded by MCIN/AEI/10.13039/501100011033.

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