Interpretations: A logician's toolkit.

Canal Ad Infinitum*
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Abstract

First-order theories are objects of undeniable interest to philosophers, logicians and mathematicians. Such theories have three central components: formal language, deductive apparatus and semantics. The formal language is a collection of symbols from which we establish the definitions of term, formula, sentence, free variable, and so on. The deductive apparatus fixes a collection of sentences and establishes the rules according to which new formulas are obtained. From this we define the concepts of axiom, rules of inference, deduction, theorem, etc. First-order structures are a very versatile instrument for fixing the semantics and with then we define, among others, the notion of a sentence be satisfied in a strutucture.

Of course, first-order theories are not just formal games. On the contrary, one of its priorities is to systematically investigate subjects of undeniable interest to logic, mathematics and philosophy.

A logician, when analyzing the deductive apparatus of a first-order theory, must answer questions concerning decidability and consistency. Furthermore, if two different logicians are investigating a given subject, it may be the case that each of them formalizes the subject under study in theories with different languages, axioms and rules of inference. In this case, in what sense can we say that the theories studied by them concern the same subject?

A similar situation occurs when two mathematicians investigate a certain mathematical content, such as arithmetic. One of the mathematicians can study arithmetic from the perspective of the standard structure $(\mathbb{N}, +, \times, 0, 1)$ while the other investigates arithmetic from the structure of finite sets provided with the membership relation $(V_{fin}, \in)$. Since the languages of these structures are distinct, they are not isomorphic and therefore it is reasonable to ask: in what sense do these mathematicians investigate the same theme?

The concepts of interpretation between structures and interpretation between theories allow us to articulate a precise answer to the questions formulated above and will be the subject matters of this mini-course. More precisely, we will present the concept of interpretation between structures and elucidate under which conditions we can say that two non-isomorphic structures are "essentially the same". From the interpretation between structures, we will approach the notion of interpretation between theories and how this notion responds to questions about decidability and consistency of theories. All the concepts presented throughout the mini-course will be accompanied by a lot of examples and applications.

Important: The mini-course will be taught in Portuguese and it is aimed at undergraduate and graduate students who have taken some course in classical logic during their undergraduate studies. The basic references on the subject are [1], [2] and [3] and examples of applications are the pappers [4], [5] and [6].

References

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