On Proof Theory in Computational Complexity

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Abstract

The subject *logic in computer science* entails proof theoretic applications. So the question arises whether open problems in computational complexity can be solved by advanced proof theoretic techniques. In particular, consider the complexity classes NP, $coNP$ and $PSPACE$. It is well-known that NP and $coNP$ are both contained in $PSPACE$, but till recently precise characterization of these relationships remained open. Now in joint papers with E. H. Haeusler [1], [2] (see also [3]) we presented proofs of the equalities $NP = coNP = PSPACE$. These results were obtained by appropriate proof theoretic tree-to-dag compressing techniques, as follows.

Recall that by conventional interpretation of ND (: *natural deductions*), derivations are rooted trees whose nodes are labeled with formulas, ordered according to the inference rules allowed; top formulas and the root formula are called assumptions and conclusion, respectively. Proofs are derivations whose all assumptions are discharged [5]. We use more liberal interpretation that allows dag-like derivations interpreted as DAGs (: *directed acyclic graphs*), not necessarily trees. Obviously dag-like derivations can be exponentially smaller than tree-like counterparts (whereas our dag-like proofs require a special notion of correctness). We elaborated a method of twofold horizontal compression of certain "huge" *quasi-polynomial* exponential-weight tree-like proofs ∂ into equivalent "small" polynomial-weight dag-like proofs ∂_0 containing only different formulas at every horizontal level, whose correctness is verifiable in polynomial time by a deterministic TM. First part of compression is defined [1] by plain deterministic recursion on the height that provides us with "small" polynomial-weight dag-like proofs in a modified ND that allows multiple-premise inferences. In the second part [2] we apply nondeterministic recursion to eliminate multiple premises and eventually arrive at "small" dag-like proofs ∂_0 in basic ND, as desired. As an application [3] we consider simple directed graphs G and canonical "huge" tree-like exponential-weight (though polynomial-height) normal deductions (derivations) ∂ whose conclusions are valid iff G have no Hamiltonian cycles. By the horizontal compression we obtain equivalent "small" polynomial-weight dag-like proofs ∂_0 and observe that the correctness of ∂_0 is verifiable in polynomial time by a deterministic TM. Since Hamiltonian Graph Problem is coNP-complete, the existence of such polynomial-weight proofs ∂_0 proves $NP = coNP$ [2], [3]. Now consider problem $NP =?PSPACE$. It is known that the validity problem in propositional minimal logic is PSPACE-complete. Moreover, minimal tautologies are provable in Hudelmaier's cutfree sequent calculus [4] by polynomial-height tree-like derivations ∂. Standard translation into ND in question yields corresponding "huge" tree-like proofs ∂' that can be horizontally compressed into desired "small" dag-like polynomial-weight proofs ∂_0 whose correctness is deterministically verifiable in polynomial time. This yields $NP = PSPACE$ [2].

References

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